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THE D_3-TRIANGULATION FOR SIMPLICIAL DEFORMATION ALGORITHMS FOR COMPUTING SOLUTIONS OF NONLINEAR EQUATIONS

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Abstract - We construct a new triangulation of continuous refinement of grid sizes of $(0, 1] \times \mathbb{R}^n$ for simplicial deformation algorithms, which is called the D_3-triangulation. We demonstrate that the D_3-triangulation is superior to the well known K_3- and J_3-triangulations when we count the number of simplices in the unit cube.

Keywords: Triangulations, Simplicial Deformation Algorithms, Measures of Efficiency of Triangulations

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1. Introduction

In order to find a fixed point or a zero point of a nonlinear mapping, several simplicial fixed point algorithms have been proposed, see [3], [4], [5] and [6]. These algorithms are based on triangulations and the complementary pivoting technique, see also [1] for a survey on this. The class of deformation algorithms, which was obtained by using triangulations with continuous refinement of grid sizes, is one of the most successful algorithms, see [3] and [5]. To improve this kind of algorithms (or reduce the cost of computation), we construct a new triangulation with continuous refinement of grid sizes of $(0, 1] \times \mathbb{R}^n$ by using the D_1-triangulation of $\mathbb{R}^n$ proposed in [2], and we show that it is superior to the well known K_3- and J_3-triangulations when counting the number of simplices.

The second section introduces the construction of the D_3-triangulation. We give the definition of the D_3-triangulation in section 3. The pivot rules of the D_3-triangulation are presented in section 4. We compare the K_3-, J_3-, and D_3-triangulations in section 5.

2. The Construction of the D_3-Triangulation

Let $w \in \mathbb{R}^n$ be such that all components of w are integers. Let $N = \{1, 2, \ldots, n\}$, $I_o(w) = \{i \in N \mid w_i \text{ is odd}\}$, and $I_e(w) = \{j \in N \mid w_j \text{ is even}\}$. Let $A(w) = \{x \in \mathbb{R}^n \mid w_i-1 \le x_i \le w_i+1 \text{ for all } i \in I_o(w);\ x_i = w_i \text{ for all } i \in I_e(w)\}$ and $B(w) = \{x \in \mathbb{R}^n \mid x_i = w_i \text{ for all } i \in I_o(w);\ w_i-1 \le x_i \le w_i+1 \text{ for all } i \in I_e(w)\}$.

Let k be a nonnegative integer. We denote $\{2^{-k}\} \times A(w)$ and

$\{2^{-(k+1)}\} \times B(w)$ by $\begin{bmatrix} 2^{-k} \\ A(w) \end{bmatrix}$ and $\begin{bmatrix} 2^{-(k+1)} \\ B(w) \end{bmatrix}$, respectively. Let

$$D(w) = \text{conv}\left\{\begin{bmatrix} 2^{-k} \\ A(w) \end{bmatrix} \cup \begin{bmatrix} 2^{-(k+1)} \\ B(w) \end{bmatrix}\right\},$$

where conv $\{.\}$ means the convex hull of some set.

<u>Lemma 2.1</u>. Let $d = (d_0, d_1, \ldots, d_n)^\top$. We have

$$D(w) = \left\{ d \in [2^{-(k+1)}, 2^{-k}] \times \mathbb{R}^n \;\middle|\; \begin{array}{l} |d_i - w_i| \le 2^{k+1}d_0 - 1 \text{ for all } i \in I_o(w) \\ \\ |d_i - w_i| \le 2 - 2^{k+1}d_0 \text{ for all } i \in I_e(w) \end{array} \right\}.$$

<u>Proof</u>. Let C denote the set in the right side of the above formula.

Let $\bar{x} = (x_0, x)^\top \in D(w)$. Then there exist $\bar{x}^A = (2^{-k}, x^A)^\top \in \begin{bmatrix} 2^{-k} \\ A(w) \end{bmatrix}$, $\bar{x}^B = (2^{-(k+1)}, x^B)^\top \in \begin{bmatrix} 2^{-(k+1)} \\ B(w) \end{bmatrix}$, $\lambda_A \ge 0$, and $\lambda_B \ge 0$ such that $\lambda_A + \lambda_B = 1$ and $\bar{x} = \lambda_A \bar{x}^A + \lambda_B \bar{x}^B$. Thus

$$x_i - w_i = (2^{k+1}x_0 - 1)(x_i^A - w_i), \text{ for all } i \in I_o(w),$$
$$x_i - w_i = (2 - 2^{k+1}x_0)(x_i^B - w_i), \text{ for all } i \in I_e(w).$$

Therefore, $\bar{x} \in C$. This means $D(w) \subseteq C$.

Let $\bar{x} = (x_0, x)^\top \in C$, $\lambda_A = 2^{k+1}x_0 - 1$, and $\lambda_B = 2 - 2^{k+1}x_0$.

If $\lambda_A = 0$ or $\lambda_B = 0$, then $\bar{x} \in \begin{bmatrix} 2^{-(k+1)} \\ B(w) \end{bmatrix}$ or $\bar{x} \in \begin{bmatrix} 2^{-k} \\ A(w) \end{bmatrix}$, respectively.

If $\lambda_A > 0$ and $\lambda_B > 0$, let x^A and x^B in $\mathbb{R}^n$ be defined by

$$x_i^A = (x_i - (2 - 2^{k+1}x_0)w_i)/(2^{k+1}x_0 - 1) \text{ and } x_i^B = w_i, \text{ for all } i \in I_o(w),$$
$$x_i^B = (x_i - (2^{k+1}x_0 - 1)w_i)/(2 - 2^{k+1}x_0) \text{ and } x_i^A = w_i, \text{ for all } i \in I_e(w).$$

Let $\bar{x}^A = (2^{-k}, x^A)^\top$ and $\bar{x}^B = (2^{-(k+1)}, x^B)^\top$. Then

$$\bar{x}^A \in \begin{bmatrix} 2^{-k} \\ A(w) \end{bmatrix} \text{ and } \bar{x}^B \in \begin{bmatrix} 2^{-(k+1)} \\ B(w) \end{bmatrix}.$$

But $\bar{x} = \lambda_A \bar{x}^A + \lambda_B \bar{x}^B$, thus $\bar{x} \in D(w)$. This means $C \subseteq D(w)$. We complete the proof.

Let $Q = \{w \in \mathbb{R}^n \mid \text{all components of } w \text{ are integers}\}$.

Lemma 2.2. $\bigcup_{w \in Q} D(w) = [2^{-(k+1)}, 2^{-k}] \times \mathbb{R}^n$.

Proof. Let $d \in [2^{-(k+1)}, 2^{-k}] \times \mathbb{R}^n$, and let $w \in \mathbb{R}^n$ be defined by

$$w_i = \begin{cases} \lfloor d_i \rfloor & \text{, if } \lfloor d_i \rfloor \text{ is even and } d_i - \lfloor d_i \rfloor \le 2 - 2^{k+1} d_0 \text{ or} \\ & \quad \lfloor d_i \rfloor \text{ is odd and } d_i - \lfloor d_i \rfloor \le 2^{k+1} d_0 - 1, \\ \lfloor d_i \rfloor + 1 & \text{, if } \lfloor d_i \rfloor \text{ is even and } d_i - \lfloor d_i \rfloor > 2 - 2^{k+1} d_0 \text{ or} \\ & \quad \lfloor d_i \rfloor \text{ is odd and } d_i - \lfloor d_i \rfloor > 2^{k+1} d_0 - 1, \end{cases}$$

for $i = 1, 2, \ldots, n$. Then $|d_i - w_i| \le 2^{k+1} d_0 - 1$ for all $i \in I_o(w)$ and $|d_i - w_i| \le 2 - 2^{k+1} d_0$ for all $i \in I_e(w)$. Thus $d \in D(w)$. We complete the proof.

Lemma 2.3. Let $w^1, w^2 \in Q$. Then $D(w^1) \cap D(w^2)$ is either empty or a common face of both $D(w^1)$ and $D(w^2)$, and

$$D(w^1) \cap D(w^2) = \operatorname{conv} \left\{ \begin{bmatrix} 2^{-k} \\ A(w^1) \cap A(w^2) \end{bmatrix} \cup \begin{bmatrix} 2^{-(k+1)} \\ B(w^1) \cap B(w^2) \end{bmatrix} \right\}.$$

Proof. Let C denote the set in the right side of the above formula. Obviously $C \subseteq D(w^1) \cap D(w^2)$.

Let $\bar{x} = (x_0, x)^T \in D(w^1) \cap D(w^2)$. Then there exist $\lambda_A^i \ge 0$, $\lambda_B^i \ge 0$, $\bar{x}_A^i = (2^{-k}, x_A^i)^T \in \begin{bmatrix} 2^{-k} \\ A(w^i) \end{bmatrix}$, and $\bar{x}_B^i = (2^{-(k+1)}, x_B^i)^T \in \begin{bmatrix} 2^{-(k+1)} \\ B(w^i) \end{bmatrix}$ such that $\lambda_A^i + \lambda_B^i = 1$ and $\bar{x} = \lambda_A^i \bar{x}_A^i + \lambda_B^i \bar{x}_B^i$, $i = 1, 2$. Thus $\lambda_A^1 = \lambda_A^2$ and $\lambda_B^1 = \lambda_B^2$.

If $\lambda_A^1 = 0$ or $\lambda_B^1 = 0$, then $\bar{x} \in C$.

If $\lambda_A^1 > 0$ and $\lambda_B^1 > 0$, then for all $i \in N$ the following cases occur:

1) If $i \in I_o(w^1)$ and $i \in I_e(w^2)$ or $i \in I_e(w^1)$ and $i \in I_o(w^2)$, then $|w_i^1 - w_i^2| = 1$. Thus $x_{Ai}^1 = x_{Ai}^2$ and $x_{Bi}^1 = x_{Bi}^2$.

2) If $i \in I_o(w^1)$ and $i \in I_o(w^2)$ or $i \in I_e(w^1)$ and $i \in I_e(w^2)$, then $|w_i^1-w_i^2| < 2$. Thus $x_{Ai}^1 = x_{Ai}^2$ and $x_{Bi}^1 = x_{Bi}^2$.

This means $\bar{x}_A^1 = \bar{x}_A^2$ and $\bar{x}_B^1 = \bar{x}_B^2$. Thus $\bar{x} \in C$. Therefore $C \supseteq D(w^1) \cap D(w^2)$. We complete the proof.

The above conclusions can also be found in [4], but without proof.

Next, we discuss how to construct the D_3-triangulation using the above conclusions and the D_1-triangulation of $\mathbb{R}^n$.

The definition of the D_1-triangulation, see [2], is as follows. Let $D_1^{0c} = \{\, x \in \mathbb{R}^n \mid \text{all components of } x \text{ are odd}\}$. Let $0 \le p \le n-1$ be an integer. Let π be a permutation of the elements of N such that $\pi(p) < \pi(p+1) < \ldots < \pi(n)$ if $p \ge 1$ and $\pi(1) < \pi(2) < \ldots < \pi(n)$ if $p = 0$. Let $s = (s_1, s_2, \ldots, s_n)^T$ be a sign vector such that $s_i \in \{-1, +1\}$ for $i = 1, 2, \ldots, n$. Let $\bar{u}^i$ be the i-th unit vector in $\mathbb{R}^n$ for $i = 1, 2, \ldots, n$. Let $y \in D_1^{0c}$. If $p = 0$, let $y^0 = y$, and $y^k = y + s_{\pi(k)}\bar{u}^{\pi(k)}$, $k = 1, 2, \ldots, n$.

If $p \ge 1$, let $y^0 = y + s$,

$$y^k = y^{k-1} - s_{\pi(k)}\bar{u}^{\pi(k)},\ k = 1, 2, \ldots, p-1, \text{ and}$$
$$y^k = y + s_{\pi(k)}\bar{u}^{\pi(k)},\ k = p, \ldots, n.$$

Then the convex hull of $y^0, y^1, \ldots, y^n$ is a simplex, which is denoted by $D_1(y, \pi, s, p)$. The D_1-triangulation of $\mathbb{R}^n$ is the set of all such simplices $D_1(y, \pi, s, p)$. Let $\bar{D}_1$ be the set of all faces of all simplices of the D_1-triangulation. We set $\alpha_0 = 1/2$ and $\alpha_{k+1} = \alpha_k/2$ for $k = 0, 1, 2, \ldots$.

For $k = 0, 1, 2, \ldots$, let

$$2\alpha_k\bar{D}_1 = \{2\alpha_k\sigma \mid \sigma \in \bar{D}_1\},$$
$$2\alpha_{k+1}\bar{D}_1 = \{2\alpha_{k+1}\sigma \mid \sigma \in \bar{D}_1\},$$
$$\alpha_k A(w) = \{\alpha_k x \mid x \in A(w)\},$$
$$\alpha_k B(w) = \{\alpha_k x \mid x \in B(w)\},$$

$$\alpha_k D(w) = \mathrm{conv}\left\{\begin{bmatrix}2^{-k}\\ \alpha_k A(w)\end{bmatrix} \cup \begin{bmatrix}2^{-(k+1)}\\ \alpha_k B(w)\end{bmatrix}\right\},$$

$$2\alpha_k\bar{D}_1 \,\Big|\, \alpha_k A(w) = \{\sigma \subseteq \alpha_k A(w) \,\Big|\, \sigma \in 2\alpha_k\bar{D}_1 \text{ and } \dim\sigma = \dim A(w)\}, \text{ and}$$

$$2\alpha_{k+1}\bar{D}_1 \,\Big|\, \alpha_k B(w) = \{\sigma \subseteq \alpha_k B(w) \,\Big|\, \sigma \in 2\alpha_{k+1}\bar{D}_1 \text{ and } \dim\sigma = \dim B(w)\}.$$

From the definition of the D_1-triangulation, it is obvious that $2\alpha_k\bar{D}_1 \,\Big|\, \alpha_k A(w)$ and $2\alpha_{k+1}\bar{D}_1 \,\Big|\, \alpha_k B(w)$ triangulate $\alpha_k A(w)$ and $\alpha_k B(w)$, respectively.

Let $s = |I_o(w)|$ and $t = |I_e(w)|$. Let

$\sigma_A = [y_A^0, y_A^1, \ldots, y_A^s] \in 2\alpha_k\bar{D}_1 \mid \alpha_k A(w)$ and

$\sigma_B = [y_B^0, y_B^1, \ldots, y_B^t] \in 2\alpha_{k+1}\bar{D}_1 \mid \alpha_k B(w)$.

<u>Lemma 2.4</u>. Let $\sigma = \mathrm{conv}\left\{\begin{bmatrix}2^{-k}\\ \sigma_A\end{bmatrix} \cup \begin{bmatrix}2^{-(k+1)}\\ \sigma_B\end{bmatrix}\right\}$. Then σ is a simplex, and

$$\sigma = \mathrm{conv}\left\{\begin{bmatrix}2^{-k}_0\\ y_A^0\end{bmatrix}, \begin{bmatrix}2^{-k}_1\\ y_A^1\end{bmatrix}, \ldots, \begin{bmatrix}2^{-k}_s\\ y_A^s\end{bmatrix}, \begin{bmatrix}2^{-(k+1)}_0\\ y_B^0\end{bmatrix}, \begin{bmatrix}2^{-(k+1)}_1\\ y_B^1\end{bmatrix}, \ldots, \begin{bmatrix}2^{-(k+1)}_t\\ y_B^t\end{bmatrix}\right\}.$$

<u>Proof</u>. Suppose that there exist $\lambda_0^A, \lambda_1^A, \ldots, \lambda_s^A, \lambda_0^B, \lambda_1^B, \ldots, \lambda_t^B$ such that

$\lambda_0^A + \lambda_1^A + \ldots + \lambda_s^A + \lambda_0^B + \lambda_1^B + \ldots + \lambda_t^B = 0,$

$$\sum_{i=0}^{s} \lambda_i^A \begin{bmatrix}2^{-k}_i\\ y_A^i\end{bmatrix} + \sum_{j=0}^{t} \lambda_j^B \begin{bmatrix}2^{-(k+1)}_j\\ y_B^j\end{bmatrix} = 0,$$

and $\lambda_0^A, \lambda_1^A, \ldots, \lambda_s^A, \lambda_0^B, \lambda_1^B, \ldots, \lambda_t^B$ are not all zeros.

Then $\sum_{i=0}^{s} \lambda_i^A = 0$, $\sum_{j=0}^{t} \lambda_j^B = 0$, $\sum_{i=0}^{s} \lambda_i^A y_A^i = 0$, and $\sum_{j=0}^{t} \lambda_j^B y_B^j = 0$.

Therefore, $y_A^0, y_A^1, \ldots, y_A^s$ or $y_B^0, y_B^1, \ldots, y_B^t$ are affinely dependent. The contradiction appears. The lemma follows immediately.

Let $T(k,k+1)$ be the set of all such simplices σ as defined in Lemma 2.4, for $k = 0, 1, 2, \ldots$.

Lemma 2.5. For any σ^1, $\sigma^2 \in T(k,k+1)$, $\sigma^1 \cap \sigma^2$ is either empty or a common face of both σ^1 and σ^2.

Proof. Let $\sigma^1 = \mathrm{conv}\left\{\begin{bmatrix}2^{-k}\\ 1\\ \tau_A\end{bmatrix} \cup \begin{bmatrix}2^{-(k+1)}\\ 1\\ \tau_B\end{bmatrix}\right\} \subsetneq \alpha_k D(w^1)$, and let

$$\sigma^2 = \mathrm{conv}\left\{\begin{bmatrix}2^{-k}\\ 2\\ \tau_A\end{bmatrix} \cup \begin{bmatrix}2^{-(k+1)}\\ 2\\ \tau_B\end{bmatrix}\right\} \subsetneq \alpha_k D(w^2).$$

From Lemma 2.3, we have $\sigma^1 \cap \sigma^2 = \mathrm{conv}\left\{\begin{bmatrix}2^{-k}\\ 1\quad 2\\ \tau_A \cap \tau_A\end{bmatrix} \cup \begin{bmatrix}2^{-(k+1)}\\ 1\quad 2\\ \tau_B \cap \tau_B\end{bmatrix}\right\}$.

But $\tau_A^1 \cap \tau_A^2$ is either empty or a common face of both τ_A^1 and τ_A^2, and $\tau_B^1 \cap \tau_B^2$ is either empty or a common face of both τ_B^1 and τ_B^2. So the lemma is correct.

Lemma 2.6. $\bigcup_{\sigma \in T(k,k+1)} \sigma = [2^{-(k+1)}, 2^{-k}] \times \mathbb{R}^n$.

Proof. From Lemma 2.2, the lemma follows immediately.

Lemma 2.7. $T(k,k+1)$ triangulates $[2^{-(k+1)}, 2^{-k}] \times \mathbb{R}^n$.

Proof. From Lemma 2.4, Lemma 2.5, and Lemma 2.6, we have this conclusion.

Theorem 2.8. $\bigcup_{k=0}^{\infty} T(k,k+1)$ triangulates $(0, 1] \times \mathbb{R}^n$.

Proof. From the choice of α_k for $k = 0, 1, 2, \ldots$, this conclusion follows from Lemma 2.7.

3. The D_3-Triangulation for Simplicial Continuous Deformation Algorithms

Let $D_3^{0c} = \{y \in (0, 1] \times \mathbb{R}^n \mid y = (y_0, y_1, \ldots, y_n)^\top$ is such that for some integer $k \geq 1$, $y_0 = 2^{-k}$ and y_i/y_0, $i = 1, 2, \ldots, n$, are all odd$\}$.

Let $N_0 = \{0, 1, 2, \ldots, n\}$ and let u^i denote the i-th unit vector in $\mathbb{R}^{n+1}$ for $i = 0, 1, 2, \ldots, n$. Let $s = (s_1, s_2, \ldots, s_n)^\top$ be a sign vector such that $s_i \in \{-1, +1\}$ for $i = 1, 2, \ldots, n$. Let $y \in D_3^{0c}$. Let

$$t_i = t_i(y) = \begin{cases} -1, & \text{if } y_i/y_0 = 1 \pmod 4, \\ +1, & \text{if } y_i/y_0 = 3 \pmod 4, \end{cases} \quad i = 1, 2, \ldots, n.$$

Let $\pi = (\pi(0), \ldots, \pi(n))$ be a permutation of the elements of N_0 and let j be such that $\pi(j) = 0$. Next, for $i = 0, 1, \ldots, j-1$, let

$$w_{\pi(i)} = (y_{\pi(i)} + y_0 s_{\pi(i)})/y_0.$$

Let $I_e = \{\pi(i) \mid 0 \leq i \leq j-1,\ w_{\pi(i)}/2 \text{ is even}\}$ and let $h = |I_e|$.
Finally, let p_1 and p_2 be integers such that $-1 \leq p_1 \leq j-2$ and $0 \leq p_2 \leq n-j-1$ if $h = 0$, and $p_2 = n-j$ if $h > 0$. Then $y^{-1}, y^0, \ldots, y^n$ are defined as follows.

<u>Definition 3.1</u>.

If $p_1 = -1$, then $y^{-1} = y$, $y^i = y + y_0 s_{\pi(i)} u^{\pi(i)}$, $i = 0, 1, \ldots, j-1$.

If $p_1 \geq 0$, then $y^{-1} = y + y_0 \sum_{i=0}^{j-1} s_{\pi(i)} u^{\pi(i)}$,

$$y^i = y^{i-1} - y_0 s_{\pi(i)} u^{\pi(i)}, \quad i = 0, 1, \ldots, p_1-1,$$
$$y^i = y + y_0 s_{\pi(i)} u^{\pi(i)}, \quad i = p_1, \ldots, j-1.$$

If $h > 0$, then
$$y^j = y + y_0 \sum_{i=0}^{j-1} s_{\pi(i)} u^{\pi(i)} + y_0 \sum_{i=j+1}^{n} t_{\pi(i)} u^{\pi(i)} + y_0 u^0,$$
$$y^i = y^{i-1} - 2y_0 t_{\pi(i)} u^{\pi(i)}, \quad i = j+1, \ldots, n.$$

If $h = 0$ and $p_2 = 0$, then

$$y^j = y + y_0 \sum_{i=0}^{j-1} s_{\pi(i)} u^{\pi(i)} - y_0 \sum_{i=j+1}^{n} t_{\pi(i)} u^{\pi(i)} + y_0 u^0,$$
$$y^i = y^j + 2y_0 t_{\pi(i)} u^{\pi(i)}, \quad i = j+1, \ldots, n.$$

Finally, if $h = 0$ and $p_2 \geq 1$, then

$$y^j = y + y_0 \sum_{i=0}^{j-1} s_{\pi(i)} u^{\pi(i)} + y_0 \sum_{i=j+1}^{n} t_{\pi(i)} u^{\pi(i)} + y_0 u^0,$$

$$y^i = y^{i-1} - 2y_0 t_{\pi(i)} u^{\pi(i)},\ i = j+1,\ \dots,\ j+p_2-1,$$

and when $p_2 = n-j-1$,

$$y^{n-1} = y^{n-2} - 2y_0 t_{\pi(n-1)} u^{\pi(n-1)},$$

$$y^n = y^{n-2} - 2y_0 t_{\pi(n)} u^{\pi(n)},$$

and when $p_2 < n-j-1$, then with

$$y^* = y + y_0 \sum_{i=0}^{j-1} s_{\pi(i)} u^{\pi(i)} - y_0 \sum_{i=j+1}^{n} t_{\pi(i)} u^{\pi(i)} + y_0 u^0,$$

$$y^i = y^* + 2y_0 t_{\pi(i)} u^{\pi(i)},\ i = j+p_2,\ \dots,\ n.$$

<u>Lemma 3.1</u>. Let $y^{-1}, y^0, \dots, y^n$ be produced from the above definition. Let $\sigma = \operatorname{conv}\{y^{-1}, y^0, \dots, y^n\}$ be their convex hull. Then σ is a simplex of $T(k,k+1)$.

<u>Proof</u>. From the definition of $y^{-1}, y^0, \dots, y^n$, we can rewrite them into

$$y^{-1} = \begin{bmatrix} y_0 \\ \bar{y}^{-1} \end{bmatrix},\ y^0 = \begin{bmatrix} y_0 \\ \bar{y}^{0} \end{bmatrix},\ \dots,\ y^{j-1} = \begin{bmatrix} y_0 \\ \bar{y}^{j-1} \end{bmatrix},$$

$$y^{j} = \begin{bmatrix} 2y_0 \\ \bar{y}^{j} \end{bmatrix},\ y^{j+1} = \begin{bmatrix} 2y_0 \\ \bar{y}^{j+1} \end{bmatrix},\ \dots,\ y^{n} = \begin{bmatrix} 2y_0 \\ \bar{y}^{n} \end{bmatrix}.$$

Let $\bar{\tau}_B = \operatorname{conv}\{\bar{y}^{-1}, \bar{y}^0, \dots, \bar{y}^{j-1}\}$ and

$\bar{\tau}_A = \operatorname{conv}\{\bar{y}^{j}, \bar{y}^{j+1}, \dots, \bar{y}^{n}\}$.

Let $w_{\pi(i)} = (y_{\pi(i)} + y_0 s_{\pi(i)})/y_0$, $i = 0, 1, \dots, j-1$, and

$w_{\pi(i)} = y_{\pi(i)}/y_0$, $i = j+1, \dots, n$.

Then $\bar{\tau}_B \in y_0\bar{D}_1 \mid y_0B(w)$ and $\bar{\tau}_A \in 2y_0\bar{D}_1 \mid y_0A(w)$.

But then $\sigma = \operatorname{conv}\left[\begin{bmatrix} y_0 \\ \bar{\tau}_B \end{bmatrix} \cup \begin{bmatrix} 2y_0 \\ \bar{\tau}_A \end{bmatrix}\right]$. This proves the lemma.

The simplex σ in Lemma 3.1 is denoted by $D_3(y, \pi, s, p_1, p_2)$. Let D_3 be the set of all such simplices $D_3(y, \pi, s, p_1, p_2)$.

Theorem 3.2. $D_3 = \bigcup_{k=0}^{\infty} T(k,k+1)$.

Proof. From the definition of the D_1-triangulation, Lemma 3.1, the construction of $T(k,k+1)$, and the definition of D_3, we have this conclusion.

Theorem 3.2 implies that D_3 is a triangulation of $(0, 1] \times \mathbb{R}^n$. It is called the D_3-triangulation.

4. The Pivot Rules of the D_3-Triangulation

Let $\sigma = [y^{-1}, y^0, y^1, \ldots, y^n] = D_3(y, \pi, s, p_1, p_2)$ be given. We wish to obtain the unique simplex $\bar{\sigma} = [\bar{y}^{-1}, \bar{y}^0, \ldots, \bar{y}^n] = D_1(\bar{y}, \bar{\pi}, \bar{s}, \bar{p}_1, \bar{p}_2)$, containing all vertices of σ except y^i. Let $j \in N_0$ such that $\pi(j) = 0$. Let $a_0 = -1$, $a_{\pi(\ell)} = s_{\pi(\ell)}$ for $0 \le \ell < j$, and $a_{\pi(\ell)} = t_{\pi(\ell)}$ for $j < \ell \le n$. We can see that $D_3(y, \pi, s, p_1, p_2)$ can be obtained from y, π, a, p_1, and p_2. Let $\bar{a}_0 = -1$, $\bar{a}_{\bar{\pi}(\ell)} = \bar{s}_{\bar{\pi}(\ell)}$ for $0 \le \ell < \bar{j} = \bar{\pi}^{-1}(0)$, and $\bar{a}_{\bar{\pi}(\ell)} = \bar{t}_{\bar{\pi}(\ell)}$ for $\bar{j} < \ell \le n$. Then $D_3(\bar{y}, \bar{\pi}, \bar{s}, \bar{p}_1, \bar{p}_2)$ can be obtained from $\bar{y}$, $\bar{\pi}$, $\bar{a}$, $\bar{p}_1$, and $\bar{p}_2$. Table (4-1) shows how $\bar{y}$, $\bar{\pi}$, $\bar{a}$, $\bar{p}_1$ and $\bar{p}_2$ depend on y, π, a, p_1, p_2 and i. From table (4-1) it is easy to obtain each vertex $\bar{y}^\ell$, $\ell = -1, 0, 1, \ldots, n$, of $\bar{\sigma}$, and in particular, its new vertex.

Table (4-1). The Pivot Rules of the D_3-Triangulation

i	j	p_1	$\|I_e\|$	p_2	$\bar{y}$	$\bar{a}$	$\bar{\pi}$	$\bar{p}_1$	$\bar{p}_2$	$\bar{j}$
-1	0				$y-y_0a$	a	$(\pi(1),\pi(2),\ldots,\pi(n),\pi(0))$	p_2-1	0	n
	1	-1			$y+2y_0a_{\pi(0)}u^{\pi(0)}$	$a-2a_{\pi(0)}u^{\pi(0)}$	π	p_1	p_2	j
	$j>1$	-1			y	a	π	p_1+1	p_2	j
	$j>0$	0			y	a	π	p_1-1	p_2	j
		$p_1\geq 1$	0	0	y	$a-2a_{\pi(0)}u^{\pi(0)}$	$(\pi(1),\pi(2),\ldots,\pi(n),\pi(0))$	p_1-1	p_2	$j-1$
				$p_2\geq 1$	y	$a-2a_{\pi(0)}u^{\pi(0)}$	$(\pi(1),\ldots,\pi(j),\pi(0),\pi(j+1),\ldots,\pi(n))$	p_1-1	p_2+1	$j-1$
			$\|I_e\|>0$		y	if $a_{\pi(0)} = t_{\pi(0)}$, $\bar{a} := a$	$(\pi(1),\pi(2),\ldots,\pi(n),\pi(0))$	p_1-1	p_2	$j-1$
					y	if $a_{\pi(0)}=-t_{\pi(0)}$, $\bar{a}:=a-2a_{\pi(0)}u^{\pi(0)}$	$(\pi(1),\ldots,\pi(j),\pi(0),\pi(j+1),\ldots,\pi(n))$	p_1-1	p_2	$j-1$

i	j	p_1	$\lvert I_e \rvert$	p_2	$\bar{y}$	$\bar{a}$	$\bar{\pi}$	$\bar{p}_1$	$\bar{p}_2$	$\bar{j}$
$0 \le i \le j-1$		-1	0	0	y	$a-2a_{\pi(i)}u^{\pi(i)}$	$(\pi(0),\ldots,\pi(i-1),\pi(i+1),\ldots,\pi(n),\pi(i))$	p_1	p_2	$j-1$
				$p_2 \ge 1$	y	$a-2a_{\pi(i)}u^{\pi(i)}$	$(\pi(0),\ldots,\pi(i-1),\pi(i+1),\ldots,\pi(j),\pi(i),\pi(j+1),\ldots,\pi(n))$	p_1	p_2+1	$j-1$
			$\lvert I_e \rvert > 0$		y	if $a_{\pi(i)} = t_{\pi(i)}$, $\bar{a} := a$	$(\pi(0),\ldots,\pi(i-1),\pi(i+1),\ldots,\pi(n),\pi(i))$	p_1	p_2	$j-1$
					y	if $a_{\pi(i)} = -t_{\pi(i)}$, $\bar{a} := a-2a_{\pi(i)}u^{\pi(i)}$	$(\pi(0),\ldots,\pi(i-1),\pi(i+1),\ldots,\pi(j),\pi(i),\pi(j+1),\ldots,\pi(n))$	p_1	p_2	$j-1$
		$0 \le p_1 < j-2$, $p_1-1<i$			y	a	$(\pi(0),\ldots,\pi(p_1-1),\pi(i),\pi(p_1),\ldots,\pi(i-1),\pi(i+1),\ldots,\pi(n))$	p_1+1	p_2	j

i	j	p_1	$\lvert I_e \rvert$	p_2	$\bar{y}$	$\bar{a}$	$\bar{\pi}$	$\bar{p}_1$	$\bar{p}_2$	$\bar{j}$
$0 \le i \le j-1$		$p_1 \ge 0$, $i < p_1 - 1$			y	a	$(\pi(0),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n))$	p_1	p_2	j
		$p_1 \ge 0$ $i = p_1 - 1$			y	a	π	$p_1 - 1$	p_2	j
$j-2$		$0 \le p_1 = j-2$			$y + 2y_0 a_{\pi(j-1)} u^{\pi(j-1)}$	$a - 2a_{\pi(j-1)} u^{\pi(j-1)}$	π	p_1	p_2	j
$j-1$					$y + 2y_0 a_{\pi(j-2)} u^{\pi(j-2)}$	$a - 2a_{\pi(j-2)} u^{\pi(j-2)}$	π	p_1	p_2	j
j	$j < n-1$		0	0	y	a	π	p_1	$p_2 + 1$	j
	$n-1$	-1			y	a	$(\pi(0),\ldots,\pi(j+1),\pi(j),\ldots,\pi(n))$	p_1	p_2	$j+1$
		$p_1 \ge 0$			y	a	$(\pi(j+1),\pi(0),\ldots,\pi(j),\pi(j+2),\ldots,\pi(n))$	$p_1 + 1$	p_2	$j+1$
	$j < n$			1	y	a	π	p_1	$p_2 - 1$	j

i	j	p_1	$\lvert I_e \rvert$	p_2	$\bar{y}$	$\bar{a}$	$\bar{\pi}$	$\bar{p}_1$	$\bar{p}_2$	$\bar{j}$
j	$j<n$	-1	0	$p_2 \geq 2$	y	$a-2a_{\pi(j+1)}u^{\pi(j+1)}$	$(\pi(0),\ldots,\pi(j+1),\pi(j),\ldots,\pi(n))$	p_1	p_2-1	$j+1$
		$p_1 \geq 0$			y	$a-2a_{\pi(j+1)}u^{\pi(j+1)}$	$(\pi(j+1),\pi(0),\ldots,\pi(j),\pi(j+2),\ldots,\pi(n))$	p_1+1	p_2-1	$j+1$
		-1	$\lvert I_e \rvert > 0$		y	$a-2a_{\pi(j+1)}u^{\pi(j+1)}$	$(\pi(0),\ldots,\pi(j+1),\pi(j),\ldots,\pi(n))$	p_1	p_2	$j+1$
		$p_1 \geq 0$			y	$a-2a_{\pi(j+1)}u^{\pi(j+1)}$	$(\pi(j+1),\pi(0),\ldots,\pi(j),\pi(j+2),\ldots,\pi(n))$	p_1+1	p_2	$j+1$
	n				$y + \frac{1}{2}y_0 a$	a	$(\pi(n),\pi(0),\ldots,\pi(n-1))$	-1	p_1+1	0

i	j	p_1	$\lvert I_e \rvert$	p_2	$\bar{y}$	$\bar{a}$	$\bar{\pi}$	$\bar{p}_1$	$\bar{p}_2$	$\bar{j}$
$j<i\le n$		-1	0	0	y	$a-2a_{\pi(i)}u^{\pi(i)}$	$(\pi(0),\ldots,\pi(j-1),\pi(i),\pi(j),\ldots,\pi(i-1),\pi(i+1),\ldots,\pi(n))$	p_1	p_2	$j+1$
		$p_1\ge 0$			y	$a-2a_{\pi(i)}u^{\pi(i)}$	$(\pi(i),\pi(0),\ldots,\pi(i-1),\pi(i+1),\ldots,\pi(n))$	p_1+1	p_2	$j+1$
				$p_2\ge 1$ $i<p_2+j-1$	y	a	$(\pi(0),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n))$	p_1	p_2	j
				$1\le p_2<n-j-1$ $i>p_2+j-1$	y	a	$(\pi(0),\ldots,\pi(p_2+j-1),\pi(i),\pi(p_2+j),\ldots,\pi(i-1),\pi(i+1),\ldots,\pi(n))$	p_1	p_2+1	j
				$p_2\ge 1$ $i=p_2+j-1$	y	a	π	p_1	p_2-1	j

i	j	p_1	$\lvert I_e \rvert$	p_2	$\bar{y}$	$\bar{a}$	$\bar{\pi}$	$\bar{p}_1$	$\bar{p}_2$	$\bar{j}$
$j<i\le n$		-1	0	$i>p_2+j-1$ $p_2=n-j-1$ $p_2\ge 1$	y	a	$(\pi(0),\ldots,$ $\pi(j-1),\pi(i),$ $\pi(j),\ldots,\pi(i-1),$ $\pi(i+1),\ldots,\pi(n))$	p_1	p_2	$j+1$
		$p_1\ge 0$			y	a	$(\pi(i),\pi(0),\ldots,$ $\pi(i-1),\pi(i+1),$ $\ldots,\pi(n))$	p_1+1	p_2	$j+1$
$j<i<n$			$\lvert I_e \rvert>0$		y	a	$(\pi(0),\ldots,$ $\pi(i+1),\pi(i)$ $\ldots,\pi(n))$	p_1	p_2	j
n	$j<n$	-1			y	a	$(\pi(0),\ldots,$ $\pi(j-1),\pi(n),$ $\pi(j),\ldots,\pi(n-1))$	p_1	p_2	$j+1$
		$p_1\ge 0$			y	a	$(\pi(n),\pi(0),\pi(1)$ $\ldots,\pi(n-1))$	p_1+1	p_2	$j+1$

5. The Comparison of Several Triangulations for Simplicial Continuous Deformation Algorithms

Let $I^n = \{x \in \mathbb{R}^n \mid 0 \le x_i \le 1,\ i = 1, 2, \ldots, n\}$.

About the definitions of the K_3- and J_3-triangulations, see [7].

<u>Theorem 5.1</u>. The number of simplices of both the K_3- and the J_3-triangulation in $[\frac{1}{2}, 1] \times I^n$ is equal to $p_n = (2^{n+1}-1)n!$.

<u>Proof</u>. See [7].

<u>Theorem 5.2</u>. The number of simplices of the D_3-triangulation in $[\frac{1}{2}, 1] \times I^n$ is equal to

$q_n = \sum_{m=0}^{n} ((2^m-1)C_n^m d_m (n-m)! + C_n^m d_m d_{n-m})$, where $d_m = m + m(m-1) + \ldots + m(m-1) \ldots 4.3 + 2$ for $m \ge 2$ and $d_0 = d_1 = 1$.

<u>Proof</u>. Let $\bar{Q} = \{w \in \mathbb{R}^n \mid w = (w_1, w_2, \ldots, w_n)^T$ such that $w_i \in \{0, 1, 2\}$ for $i = 1, 2, \ldots, n\}$. Let $w \in \bar{Q}$. Let

$$\bar{A}(w) = \{x \in \mathbb{R}^n \mid w_i-1 \le x_i \le w_i+1 \text{ for each } i \in I_o(w) \text{ and } x_i = w_i \text{ for each } i \in I_e(m)\},$$

$$\bar{B}(w) = \{x \in \mathbb{R}^n \mid x_i = w_i \text{ for each } i \in I_o(w), \text{ and } w_i-1 \le x_i \le w_i \text{ if } w_i = 2, \text{ and } w_i \le x_i \le w_i+1 \text{ if } w_i = 0\},$$

and let

$$\tfrac{1}{2}\bar{D}(w) = \text{conv}\left\{\begin{bmatrix} 1 \\ \frac{1}{2}\bar{A}(w) \end{bmatrix} \cup \begin{bmatrix} 2^{-1} \\ \frac{1}{2}\bar{B}(w) \end{bmatrix}\right\}.$$

We can easily demonstrate that $\bigcup_{w \in \bar{Q}} \frac{1}{2}\bar{D}(w) = [\frac{1}{2}, 1] \times I^n$.

Let $m = |I_e(w)|$. It is obvious that there are $2^m C_n^m$ elements in $\bar{Q}$ such that m components of each of them are even. From the number of simplices in I^n of the D_1-triangulation (see [2]) and the construction of the D_3-triangulation, we know that the number of simplices of the D_3-triangulation in $\cup\{\frac{1}{2}\bar{D}(w) \mid w \in \bar{Q}$ and $|I_e(w)| = m\}$ is equal to

$$(2^m-1)C_n^m d_m (n-m)! + C_n^m d_m d_{n-m}.$$

Since $[\frac{1}{2}, 1] \times I^n = \bigcup_{m=0}^{n} (\cup\{\frac{1}{2}\bar{D}(w) \mid w \in \bar{Q}$ and $|I_e(w)| = m\})$, this proves the theorem.

Theorem 5.3. If $n \geq 3$, then $q_n < p_n$. As n goes to infinity, q_n/p_n goes to e-2.

Proof. Since, if $n \geq 3$, for $m = 0, 1, 2, \ldots, n$,

$$(2^m-1)C_n^m d_m (n-m)! + C_n^m d_m d_{n-m} < 2^m C_n^m m!(n-m)!,$$

the first part of the theorem is correct. Further,

$$q_n/p_n = (\sum_{m=0}^{n} 2^m d_m/m!)/(2^{n+1}-1) + (\sum_{m=0}^{n} ((d_{n-m}/(n-m)!)-1)d_m/m!)/(2^{n+1}-1)$$

and $d_n/n!$ goes to e-2 as n goes to infinity. Hence the second part of the theorem is correct.

From Theorem 5.3, we see that the number of simplices of the D_3-triangulation in $[\frac{1}{2}, 1] \times I^n$ is the smallest of the K_3-, J_3-, and D_3-triangulations for simplicial continuous deforming algorithms.

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